Exercise 63

Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

$$y = (3-x)(1+x)^2(1-x)^4$$

Solution

To find the y-intercept, plug in $x = 0$ to the function.

$$y = (3-0)(1+0)^2(1-0)^4 = 3$$

Therefore, the y-intercept is $(0, 3)$. To find the x-intercept(s), set $y = 0$ and solve the equation for x.

$$(3-x)(1+x)^2(1-x)^4 = 0$$

$$x = 3 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 1$$

Therefore, the x-intercepts are $(3, 0)$ and $(-1, 0)$ and $(1, 0)$. Calculate the limit of the function as $x \to \pm\infty$.

$$\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} (3-x)(1+x)^2(1-x)^4 \\
&= \lim_{x\to\infty} x\left(\frac{3}{x} - 1\right) \cdot x^2\left(\frac{1}{x} + 1\right)^2 \cdot x^4\left(\frac{1}{x} - 1\right)^4 \\
&= \lim_{x\to\infty} x^7\left(\frac{3}{x} - 1\right)\left(\frac{1}{x} + 1\right)^2\left(\frac{1}{x} - 1\right)^4 \\
&= \lim_{x\to\infty} \frac{\left(\frac{3}{x} - 1\right)\left(\frac{1}{x} + 1\right)^2\left(\frac{1}{x} - 1\right)^4}{\frac{1}{x^7}} \\
&= \frac{(0-1)(0+1)^2(0-1)^4}{0} \\
&= -\infty
\end{aligned}$$

In the second limit, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\begin{aligned}
\lim_{x\to-\infty} y &= \lim_{u\to\infty} (3+u)(1-u)^2(1+u)^4 \\
&= \lim_{u\to\infty} u\left(\frac{3}{u}+1\right)\cdot u^2\left(\frac{1}{u}-1\right)^2\cdot u^4\left(\frac{1}{u}+1\right)^4 \\
&= \lim_{u\to\infty} u^7\left(\frac{3}{u}+1\right)\left(\frac{1}{u}-1\right)^2\left(\frac{1}{u}+1\right)^4 \\
&= \lim_{u\to\infty} \frac{\left(\frac{3}{u}+1\right)\left(\frac{1}{u}-1\right)^2\left(\frac{1}{u}+1\right)^4}{\frac{1}{u^7}} \\
&= \frac{(0+1)(0-1)^2(0+1)^4}{0} \\
&= \infty
\end{aligned}$$

Below is a graph of the function versus x.